FINAL EXAM(Back Paper) : DIFFERENTIAL EQUATIONS

Duration: 3 hours **Total Marks: 50**

Answer any FIVE questions out of the given six. Each carries ten marks.

1. State Picard's theorem. Show that the initial value problem $y' = y^{\frac{2}{3}}$, $y(0) = 0$ admits infinitely many solutions.

2. Let $x_1 : [0, \alpha) \to \mathbb{R}$ and $x_2 : [0, \alpha) \to \mathbb{R}$ be the two solutions to the IVP

$$\frac{dx}{dt}(t) = (1 - x(t))x(t)$$

and $x_1(0) = a$, $x_2(0) = b$. If $a > b > 0$ then show that $x_1(t) > x_2(t)$ for all $t \in [0, \alpha)$.

3. Let $f(x)$ be a continuous function on $\mathbb{R}$. Find the general solution of the ODE

$$y'' - xf(x)y' + f(x)y = 0, \quad x \in (0, \infty).$$

[Hint: Guess one solution first]

4. Consider the differential equation $L(y) := y'' + p(x)y' + q(x)y = 0$, where p and q are continuous functions on some interval (a, b) containing the point x_0. If ψ_1 is a non-trivial solution of $L(y) = 0$ on (a, b) then show that there is a second solution ψ_2 on (a, b) such that $W(\psi_1, \psi_2)(x_0) = 1$.

5. Prove that problem

$$\begin{cases} u_x + u_y = 0 & \text{in } \mathbb{R}^2 \\ u(x, y) = g(x, y) & \text{for } (x, y) \in \Gamma. \end{cases}$$

has a unique solution when Γ is a line in $\mathbb{R}^2$ with slope not equal to one and g is a C^1 function.

Next suppose that $\Gamma = \{(x, x) : x \in \mathbb{R}\}$. Then prove that the above linear PDE admits a solution only if g is a constant on Γ. Also prove that if g is a constant, then the PDE admits infinitely many solutions.

6. Derive D'Alembert formula for the wave equation

$$\begin{aligned} &u_{tt} = u_{xx} && (x, t) \in \mathbb{R} \times (0, \infty) \\ &u(x, 0) = f(x), u_t(x, 0) = g(x) && x \in \mathbb{R}. \end{aligned}$$